8 Wavelets (2)

Monday, March 2, 2020 10:42 PM

Since it came up last class that not everyone is familiar with Nyquist, let's start with a brief detour into signal processing.

Nyquist-Shannon sampling theorem

[Kotelnikov, 1933] [Nyquist, 1928]
[E.T. Whitaker, 1915] [Shannon, 1948]
[J.M. Whitaker, 1935] [Gabor, 1946]

If a function $x(t)$ contains no frequencies B Hz or higher, it is completely determined by giving its ordinates at a series of points spaced $\frac{1}{2B}$ seconds apart.

proof. Let $X(\omega)$ be the spectrum of $x(t)$.

Then $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}\,d\omega = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} X(\omega)e^{i\omega t}\,d\omega$$

because $X(\omega) = 0$ outside the band $\left|\frac{\omega}{2\pi}\right| < B$.

Let $t = \frac{n}{2B}$, for $n \in \mathbb{Z}$.

Then $$\underbrace{x\left(\frac{n}{2B}\right)}_{\substack{\text{Sampling } x \text{ at} \\ t = \frac{n}{2B}}} = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} X(\omega)e^{i\omega\cdot\frac{n}{2B}}\,d\omega.$$

Recall the Fourier series of a periodic function $f(y)$ is given by

$$f(y) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{i\cdot\frac{2\pi n y}{P}},$$ where P is the period of $f(y)$

$$c_n = \frac{1}{P}\int_P f(y)\,e^{-i\cdot\frac{2\pi n y}{P}}.$$

Let $P = 4\pi B$, $$c_n = \frac{1}{4\pi B}\int_{-2\pi B}^{2\pi B} f(y)\,e^{-iy\cdot\frac{n}{2B}}\,dy.$$

Thus, the RHS above is precisely coefficients of the Fourier series expansion of $X(\omega)$ when taken as a $4B$-periodic function.

Now, the RHS above is precisely coefficients of the Fourier series expansion of $X(\omega)$ when taken as a $4B$-periodic function.

$\Rightarrow X(\omega)$ is completely determined by the sampling at $t = \frac{n}{2B}$, $n \in \mathbb{Z}$; for $\omega \in [-2B, 2B]$, and $X(\omega) = 0$ outside that interval.

Hence, we know $X(\omega)$ after sampling.

But $x(t)$ is completely determined by its Fourier transform $X(\omega)$.

$\Rightarrow x(t)$ is completely determined by taking samples spaced $\frac{1}{2B}$ sec apart.

Not proved here: Let x_n be the nth sample.

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \frac{\sin[\pi(2Bt-n)]}{\pi(2Bt-n)}.$$

Wavelet systems

The Haar wavelet was built from a scale function $\phi(x)$ that satisfies the dilation equation $\phi(x) = \phi(2x) + \phi(2x-1)$.

Consider a scale function solving a more general dilation equation.

$$\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x-k).$$

And let
$$\phi_{jk}(x) = \phi(2^j x - k).$$

Let $V_j = \text{span}\{\phi_{jk}\}_{k\in\mathbb{Z}}$. Then $\phi_{jk} \in V_{j+1}$, so $V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$

So we still have the nice property where each successive set of finer-grained ϕ_{jk} spans the coarser-resolution span before it.

We will show that it is in general possible to build a wavelet system of orthonormal bases out of a scaling function.

$$\sum^{2}$$

$$\sum_{k=0}^{2}$$

Solving a dilation equation

Easy to check a solution $\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x-k)$.

Harder to find a solution.

$[0, \frac{d}{2}]$

$[\frac{1}{2}, \frac{d+1}{2}]$

$\vdots$

$[\frac{d}{2}, d]$

Lemma: $\phi(x)$ has support $[0, d-1]$ (if it has finite support)

proof. Say $\phi(x)$ has finite support $[A, B]$.

Then $\phi(2x)$ has support $[\frac{A}{2}, \frac{B}{2}] \subseteq [A, B]$.

$\Rightarrow A = 0$.

Then $\phi(2x-(d-1))$ has support $[\frac{d-1}{2}, \frac{d-1}{2} + \frac{B}{2}] \subseteq [0, B]$

$\Rightarrow B = d-1$.

Note, $\phi(2x-k)$ has support $[\frac{k}{2}, \frac{d-1}{2} + \frac{k}{2}]$.

Cascade algorithm

[Malone, PhD thesis, 2000]

Define an operator $\mathcal{V}$ by

$$(\mathcal{V}f)(x) = \sum_{k=0}^{d-1} c_k f(2x-k).$$

Then we clearly are looking for a fixed point of this operator.

Let f be a compactly supported solution on $[0, N]$. i.e. $\sum_{k=0}^{N} c_k f(2x-k)$

Then
$$\left.\begin{aligned}
(\mathcal{V}f)(0) &= c_0 f(0) \\
(\mathcal{V}f)(1) &= c_2 f(0) + c_1 f(1) + c_0 f(2) \\
(\mathcal{V}f)(2) &= c_4 f(0) + c_3 f(1) + c_2 f(2) + c_1 f(3) + c_0 f(4) \\
&\vdots \\
(\mathcal{V}f)(N-1) &= c_N g(N-2) + c_{N-1} g(N-1) + c_{N-2} g(N) \\
(\mathcal{V}f)(N) &= c_N g(N)
\end{aligned}\right\}$$

Because of the supports of each $f(2x-k)$.

So let $\vec{g} = [f(0), f(1), \ldots, f(N)]^T$

Then $\vec{g} = M\vec{g}$, where the matrix M is the given by $(\mathcal{V}f)$ above.

$$M = \begin{bmatrix} c_0 & & & & & & & \\ c_2 & c_1 & c_0 & & & 0 & & \\ c_4 & c_3 & c_2 & c_1 & c_0 & & & \\ & & & & \ddots & & & \\ & & 0 & & & c_N & c_{N-1} & c_{N-2} \\ & & & & & & & c_N \end{bmatrix}$$

M under certain conditions has a nice spectral gap, so this iteration converges rapidly to an approximation $\vec{g}$ of $f(x)$.

Alternate solution by increasing resolution

Consider $\phi(x) = \frac{1}{2} f(2x) + f(2x-1) + \frac{1}{2} f(2x-2)$ with support $[0, 2)$.

$$\phi(0) = \frac{1}{2}\phi(0) + \phi(-1) + \frac{1}{2}\phi(-2) = \frac{1}{2}\phi(0) + 0 + 0 \Rightarrow \phi(0) = 0$$

$$\phi(2) = \frac{1}{2}\phi(4) + \phi(3) + \frac{1}{2}\phi(2) = \frac{1}{2}\phi(2) \Rightarrow \phi(2) = 0$$

$$\phi(1) = \frac{1}{2}\phi(2) + \phi(1) + \frac{1}{2}\phi(0) = \phi(1). \Rightarrow \phi(1) \text{ arbitrary.}$$

Makes sense because you can always scale.

Set $\phi(1) = 1$.

Then

$$\phi\left(\tfrac{1}{2}\right) = \frac{1}{2}\phi(1) + \phi(0) + \frac{1}{2}\phi(-1) = \frac{1}{2}$$

$$\phi\left(\tfrac{3}{2}\right) = \frac{1}{2}\phi(3) + \phi(2) + \frac{1}{2}\phi(1) = \frac{1}{2}$$

$$\phi\left(\tfrac{1}{4}\right) = \frac{1}{2}\phi\left(\tfrac{1}{2}\right) + \phi\left(-\tfrac{1}{2}\right) + \frac{1}{2}\phi\left(-\tfrac{3}{2}\right) = \frac{1}{4}$$

$$\vdots$$

Can compute any $\phi\left(\frac{i}{2^j}\right)$ for larger values of j until we get desired accuracy.
If $\phi(x)$ is simple, can then conjecture explicit form.

Conditions on the dilation equation

Conditions on the dilation equation

(Recall)

Lemma 11.1. If $\phi(x) = \sum_{k=1}^{d-1} c_k \,(2x-k)$, then either $\sum_{k=0}^{d-1} c_k = 2$ or $\int_{-\infty}^{\infty} \phi(x)\,dx = 0$.

Lemma 11.2. Let $\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x-k)$. If $\phi(x)$ and $\phi(x-k)$ are orthogonal for $k \neq 0$ and $\phi(x)$ has been normalized so $\int_{-\infty}^{\infty} \phi(x)\phi(x-k)\,dx = \delta(k)$, then $\sum_{i=0}^{d-1} c_i c_{i-2k} = 2\,\delta(k)$.

Where $\delta(x)$ is the Kronecker delta $\delta(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$

proof.

$$\delta(k) = \int_{-\infty}^{\infty} \phi(x)\,\phi(x-k)\,dx = \int_{-\infty}^{\infty} \left(\sum_{i=0}^{d-1} c_i \phi(2x-i) \right)\left(\sum_{j=0}^{d-1} c_j \,\phi(2x-2k-j) \right) dx$$

$$= \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} c_i c_j \int_{-\infty}^{\infty} \phi(2x-i)\,\phi(2x-2k-j)\,dx$$

$$= \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} c_i c_j \,\frac{1}{2} \int_{-\infty}^{\infty} \phi(x-i)\,\phi(x-2k-j)\,dx$$

$$= \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} c_i c_j \,\frac{1}{2} \int_{-\infty}^{\infty} \phi(x)\,\phi(x+i-2k-j)\,dx$$

$$= \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} c_i c_j \cdot \frac{1}{2}\, \delta(2k+j-i)$$

$$= \frac{1}{2} \sum_{i=0}^{d-1} c_i c_{i-2k}$$

$$\Rightarrow \quad \sum_{i=0}^{d-1} c_i c_{i-2k} = 2\,\delta(k). \qquad \blacksquare$$

This is a necessary but not sufficient condition for orthogonality.

Lemma 11.3 If $[0, d-1)$ is the support of $\phi(x)$, and the set of integer shifts, $\{\phi(x-k)\}, k \geq 0\}$ are linearly independent, then $c_k = 0$ unless $0 \leq k \leq d-1$

proof. $\phi(x) = \sum_{k=-\infty}^{\infty} c_k \phi(2x-k)$. Note because $\phi(2x-k)$ are lin. ind., the c_k are unique

If the support of $\phi(x)$ is $[0, d-1)$, then support of $\phi(2x)$ is $[0, \frac{d-1}{2})$.

So support of $\phi(2x-k)$ is $[\frac{k}{2}, \frac{k}{2} + \frac{d-1}{2})$

The support of the LHS and RHS have to be the same.

But since $\phi(2x-k)$ are linearly ind., $c_k \phi(2x-k)$ for $k \geq d$ cannot be "canceled out" by other terms in the summation if $c_k > 0$.

So, $$\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x-k)$$

Lemma: d is even.

Suppose d is odd.

By Lemma 11.2, $\sum_{i=0}^{d-1} c_i c_{i-2k} = 0$ for $k \neq 0$.

Let $k = \frac{d-1}{2}$. Then $0 = \sum_{i=0}^{d-1} c_i c_{i-d+1} = c_{d-1} c_0 \Rightarrow c_0$ or c_{d-1} is 0.

Then there are only $d-1$ non-zero coefficients, so we can just shift so that d is even.

Thus, If a dilation equation has d terms with

$$\sum_{k=0}^{d-1} c_k = 2, \quad \sum_{i=0}^{d-1} c_i c_{i-2k} = 2\delta(k) \text{ for } 1 \leq k \leq \frac{d-1}{2}, \text{ then for } d > 2,$$

there are $\frac{d}{2} - 1$ coefficients we can use to define a wavelet system.

Note that these are still not sufficient conditions for an orthonormal basis, just a set of nice scale functions.

Derivation of wavelets from scaling function

Let the "mother wavelet" $\psi(x) = \sum_{k=0}^{d-1} b_k \phi(2x-k)$.

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We want integer shifts of $\psi(x)$ to be orthogonal to each other and to the scaling function $\phi(x)$.

Lemma 11.5 (orthogonality of $\phi(x)$ and $\psi(x-k)$)

Let $\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x-k)$ and $\psi(x) = \sum_{k=0}^{d-1} b_k \phi(2x-k)$.

If $\int_{-\infty}^{\infty} \phi(x)\phi(x-k)\,dx = \delta(k)$ and $\int_{-\infty}^{\infty} \phi(x)\psi(x-k)\,dx = 0$ for all k,

then $\sum_{i=0}^{d-1} c_i b_{i-2k} = 0$ for all k.

proof.

$$\int_{-\infty}^{\infty} \phi(x)\psi(x-k)\,dx = \int_{-\infty}^{\infty} \sum_{i=0}^{d-1} c_i \phi(2x-i) \sum_{j=1}^{d-1} b_j \phi(2x-2k-j)\,dx = 0$$

$$\Rightarrow \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} c_i b_j \int_{-\infty}^{\infty} \phi(2x-i)\,\phi(2x-2k-j)\,dx = 0$$

$$\Rightarrow \frac{1}{2}\sum_{i=0}^{d-1}\sum_{j=0}^{d-1} c_i b_j \int_{-\infty}^{\infty} \phi(y)\,\phi(y-2k-j+i)\,dy = 0 \qquad (y = 2x-i)$$

$$\Rightarrow \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} c_i b_j\, \delta(2k+j-i) = 0$$

$$\Rightarrow \sum_{i=0}^{d-1} c_i b_{i-2k} = 0$$

$\square$

Lemma 11.4 Let $\psi(x) = \sum_{k=0}^{d-1} b_k \phi(2x-k)$. If $\psi(x)$ and $\psi(x-k)$ are orthogonal for $k \neq 0$ and $\psi(x)$ has been normalized so $\int_{-\infty}^{\infty} \psi(x)\psi(x-k)\,dx = \delta(k)$, then $\sum_{i=0}^{d-1} b_i b_{i-2k} = 2\delta(k)$

proof. Analogous to Lemma 11.2.

Lemma 11.6 If $\langle \phi(x), \phi(x-k)\rangle = \delta(k)$

$\langle \phi(x), \psi(x-k)\rangle = 0$ for all k,

then $b_k = (-1)^k c_{d-1-k}$.

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proof. By Lemma 11.5 $\sum_{j=0}^{d-1} c_j b_{j-2k} = 0$ for all k.

$$\Rightarrow \sum_{j=0}^{\frac{d}{2}-1} c_{2j} b_{2j-2k} + \sum_{j=0}^{\frac{d}{2}-1} c_{2j+1} b_{2j+1-2k} = 0.$$

By lemmas 11.2 and 11.4, $\sum_{j=0}^{d-1} c_j c_{j-2k} = 2\delta(k)$ and $\sum_{j=0}^{d-1} b_j b_{j-2k} = 2\delta(k)$

$$\Rightarrow \sum_{j=0}^{\frac{d}{2}-1} c_{2j} c_{2j-2k} + \sum_{j=0}^{\frac{d}{2}-1} c_{2j+1} c_{2j+1-2k} = 2\delta(k)$$

$$\sum_{j=0}^{\frac{d-1}{2}} b_{2j} b_{2j-2k} + \sum_{j=0}^{\frac{d}{2}-1} b_{2j+1} b_{2j+1-2k} = 2\delta(k)$$

Let $C_e = (c_0, c_2, \dots, c_{d-2})$ $\quad B_e = (b_0, b_2, \dots, b_{d-2})$

$C_o = (c_1, c_3, \dots, c_{d-1})$ $\quad B_o = (b_1, b_3, \dots, b_{d-1})$

$$C_e * B_e^R = \sum_{j=0}^{\frac{d}{2}-1} c_{2j} b_{2j} \qquad C_o * B_o^R = \sum_{j=0}^{\frac{d}{2}-1} c_{2j+1} b_{2j+1}$$

$$B_o * B_o^R = \sum_{j=0}^{\frac{d}{2}-1} b_{2j+1} b_{2j+1}$$

$$\begin{pmatrix} C_e & C_o \\ B_e & B_o \end{pmatrix} * \begin{pmatrix} C_e^R & B_e^R \\ C_o^R & B_o^R \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Let $C_{e,k} = (c_{2k}, \dots, c_{d-2})$ $\quad B_{e,k} = (b_{2k}, \dots, b_{d-2})$

$C_{o,k} = (c_{2k+1}, \dots, c_{d-1})$ $\quad B_{o,k} = (b_{2k+1}, \dots, b_{d-1})$

$C_{e,k}^R = (c_{d-2-2k}, \dots, c_0)$ $\quad \dots$

Then

$$\begin{pmatrix} C_{e,k} & C_{o,k} \\ B_{e,k} & B_{o,k} \end{pmatrix} * \begin{pmatrix} C_{e,k}^R & B_{e,k}^R \\ C_{o,k}^R & B_{o,k}^R \end{pmatrix} = \begin{pmatrix} 2\delta(k) & 0 \\ 0 & 2\delta(k) \end{pmatrix}$$

Take a discrete Fourier transform (thinking of this as functions of k)

$$\begin{pmatrix} F(C_{e,k}) & F(C_{o,k}) \end{pmatrix} \begin{pmatrix} F(C_{e,k}^R) & F(B_{e,k}^R) \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \end{pmatrix}$$

Take a Fourier transform (thinking of this as functions of k)

$$\begin{pmatrix} F(C_{e,k}) & F(C_{o,k}) \\ F(B_{e,k}) & F(B_{o,k}) \end{pmatrix} \begin{pmatrix} F(C_{e,k}^R) & F(B_{e,k}^R) \\ F(C_{o,k}^R) & F(B_{o,k}^R) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Taking a determinant,

$$\Big(F(C_{e,k})F(B_{o,k}) - F(B_{e,k})F(C_{o,k})\Big)\Big(F(C_{o,k}^R)F(B_{o,k}^R) - F(C_{o,k}^R)F(B_{e,k}^R)\Big) = 4$$

$$\Rightarrow \quad F(C_{e,k})\,F(B_{o,k}) - F(C_{o,k})\,F(B_{e,k}) = 2$$

$$\Rightarrow \quad C_{e,k} * B_{o,k} - C_{o,k} * B_{e,k} = 2\,\delta(k)$$

Dropping the 'k''s for notational convenience

Convolution by C_e^R yields

$$C_e^R * C_e * B_o - C_e^R * B_e * C_o = C_e^R * 2\,\delta(k)$$

Note $-C_e^R * B_e = C_o^R * B_o$

$$\Rightarrow \quad C_e^R * C_e * B_o + C_o^R * B_o * C_o = 2\,C_e^R * \delta(k)$$

$$(C_e^R * C_e + C_o^R * C_o) * B_o = 2\,C_e^R * \delta(k)$$

$$2\,\delta(k) * B_o = 2\,C_e^R * \delta(k)$$

$$C_e^R = B_o$$

$$\Rightarrow \quad c_i = b_{d-1-i} \quad \text{for even } i.$$

Similarly, convolution by C_o^R yields

$$-B_e = C_o^R$$

$$\Rightarrow \quad c_i = -b_{d-1-i} \quad \text{for odd } i.$$

$$\Rightarrow \quad c_i = (-1)^i \cdot b_{d-1-i} \quad \text{for all } i.$$